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# A SOLUTION TO A PROBLEM OF TEODOR PRZYMUSIŃSKI

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A subset $A$ of a space $X$ is *$C^*$-embedded* in $X$ if every bounded real-valued continuous function on $A$ is continuously extendable to the whole of $X$. If this holds for all real-valued continuous functions on $A$, then $A$ is *$C$-embedded* in $X$.

The present note provides detailed suggestions to the solution of the following problem. For a *non-discrete metric* space $M$ and a subset $A$ of a space $X$, does the $C^*$-embedding of $A \times M$ in $X \times M$ imply that it is also $C$-embedded in $X \times M$, i.e.

$$A \times M \overset{C^*}{\hookrightarrow} X \times M \qquad \Longrightarrow \qquad A \times M \overset{C}{\hookrightarrow} X \times M \ \ ?$$

The problem was stated as Problem 3 of [T. Przymusiński, *Notes on extendability of continuous functions from products with a metric factor*, unpublished note, May 1983], later on as Problem 4.14 of [T. Hoshina, *Extensions of mappings II*, Topics in General Topology (K. Morita and J. Nagata, eds.), North-Holland, Amsterdam, 1989, pp. 41–80] and Problem 3.1 of [T. Hoshina, *Extensions of mappings*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), North-Holland, Amsterdam, 1992, pp. 405–416].

## THE SOLUTION

To state the main result we call in use also the following imbedding-like properties. Let $\lambda$ be an infinite cardinal number.

*$P^\lambda$-embedding*: A subset $A$ of a space $X$ is *$P^\lambda$-embedded* in $X$, or briefly $A \overset{P^\lambda}{\hookrightarrow} X$, if every continuous $f : A \to Y$ in a Banach space $Y$ of $w(Y) \leq \lambda$ is continuously extendable to the whole of $X$.

*$U^\omega$-embedding*: A subset $A$ of a space $X$ is *$U^\omega$-embedded* in $X$, or briefly $A \overset{U^\omega}{\hookrightarrow} X$, if for every continuous $f : A \to \mathbb{R}$ there exists a continuous $g : X \to \mathbb{R}$ with $f(x) \leq g(x)$ whenever $x \in A$.

It should be mentioned that $A$ is $C$-embedded in $X$ if and only if it is $P^\omega$-embedded in $X$, while $A$ is $P^\omega$-embedded in $X$ if and only if it is both $U^\omega$- and $C^*$-embedded in $X$. That is, always

$$C = P^\omega = U^\omega + C^*.$$

The following recent result was obtained together with Haruto Ohta.

**Theorem.** *For a $P^\lambda$-embedded subset $A$ of a space $X$ and a metric space $M$, the following conditions are equivalent*

(a) $A \times M \overset{P^\lambda}{\hookrightarrow} X \times M$

(b) $A \times M \overset{C^*}{\hookrightarrow} X \times M$

(c) $A \times M \overset{U^\omega}{\hookrightarrow} X \times M$

Note that $A \times M \overset{C^*}{\hookrightarrow} X \times M$ implies $A \overset{C}{\hookrightarrow} X$ provided $M$ is non-discrete because, in this case, $M$ contains an infinite compact subset. Hence, the above result provides a complete positive solution to the problem of interest. For the proper understanding of this theorem, a word should be said also about the last condition (c). The statement that it is equivalent to the previous ones should be compared with Rudin-Starbird's result that, for a non-discrete metric space $M$, the normality of $X \times M$ implies the countable paracompactness of $X \times M$. Namely, the $U^\omega$-embedding has a quite nice and useful reading just in terms of Ishikawa's characterization of countable paracompactness.

## ON THE WAY TO THE PROOF

Special cases of (a) $\Leftrightarrow$ (b): $X \times M$ an *$M$-independent* product and $\lambda = \omega$ (Przymusiński, 1983); $M = \mathbb{P}$ the space of irrational numbers and $\lambda = \omega$ (Ohta, 1993); $M$ – $\sigma$-locally compact (Yamazaki, 1997); $M^2$ homeomorphic to $M$ (Hoshina and Yamazaki, 199?).

## FIRST STEP: A reduction to "nice" metric factors

For a space $Y$, let $\mathcal{P}(Y)$ be the set of all closed subsets of $Y$. Let $A$, $X$ and $M$ be as in our theorem. To $M$ we associate the family of *all solutions*, or the *Przymusiński* family for $M$, by

$$\mathfrak{P} = \{S \subset M : A \times S \overset{P^\lambda}{\hookrightarrow} X \times S\}.$$

The following important fact will play a central role in this part of the proof.

**Fact 1** (Michael). $S \in \mathfrak{P} \implies \mathcal{P}(S) \subset \mathfrak{P}$.

It will be useful to illustrate the idea first on a partial case. For the purpose, let $M^{(\mathcal{K},0)} = M$, and, for every ordinal $\alpha > 0$, let

$$M^{(\mathcal{K},\alpha)} = X \backslash \bigcup \{K \subset M \text{ compact} : K \subset M^{(\mathcal{K},\beta)} \text{ is open for some } \beta < \alpha\}.$$

Take an ordinal $\gamma$ with $M^{(\mathcal{K},\gamma)} = M^{(\mathcal{K},\gamma+1)}$. Then,

1. $M^{(\mathcal{K},\gamma)} \in \mathcal{P}(M)$ is **nowhere locally compact**;

2. $M \backslash M^{(\mathcal{K},\gamma)}$ is $\sigma$-**locally compact**.

Now, suppose that $M$ is a Polish space with $\dim(M) = 0$. Then, relaying on the known partial solution and Fact 1, we get the following series of implications.

$$M^{(\mathcal{K},\gamma)} = \emptyset \quad \Longrightarrow \quad M \text{ is } \sigma\text{-locally compact} \quad \Longrightarrow \quad M \in \mathfrak{P}.$$

On the other hand,

$$\begin{array}{ccc} M^{(\mathcal{K},\gamma)} \neq \emptyset & \Longrightarrow & M^{(\mathcal{K},\gamma)} = \mathbb{P} \\ & & \Downarrow \\ & & M^{(\mathcal{K},\gamma)} \in \mathfrak{P} \\ & & \Downarrow \\ & & M \in \mathcal{P}(\mathbb{P}) = \mathcal{P}\left(M^{(\mathcal{K},\gamma)}\right) \subset \mathfrak{P}. \end{array}$$

That is, always $M \in \mathfrak{P}$.

Let $\mathcal{K} = \{S \in \mathcal{P}(M) : S \text{ is compact}\}$. Then, by the known results, $\mathcal{K} \subset \mathfrak{P}$. On the other hand, $M^{(\mathcal{K},\gamma)}$ is a resulting set by a $\mathcal{K}$-*scattered* procedure and, hence, a procedure that is *scattered* also with respect to a part of the members of $\mathfrak{P}$. This arguments suggest that, for a better result, we need to call in use all members of $\mathfrak{P}$, i.e. to arrange a $\mathfrak{P}$-**scattered** procedure on $M$.

Turning to this case, we change our definition as follows. Let $S \subset M$, and let $S^{(\mathfrak{P},0)} = S$. Next, for any ordinal $\alpha > 0$, we consider the set

$$S^{(\mathfrak{P},\alpha)} = S \backslash \bigcup \left\{U \subset S : U \text{ is open and } \mathrm{cl}_S(U) \cap S^{(\mathfrak{P},\beta)} \in \mathfrak{P} \text{ for some } \beta < \alpha\right\}.$$

Suppose that $M \notin \mathfrak{P}$, and let $S \in \mathcal{P}(M) \backslash \mathfrak{P}$ be such that

$$w(S) = \min\{w(F) : F \in \mathcal{P}(M) \backslash \mathfrak{P}\}.$$

Then, as before, take an ordinal $\gamma$ with $S^{(\mathfrak{P},\gamma)} = S^{(\mathfrak{P},\gamma+1)}$. As a result, we get that

1. $S^{(\mathfrak{P},\gamma)} \in \mathcal{P}(S)$ is **weight-homogeneous**, that is, $w(U) = w(S)$ for every non-empty open $U \subset S$;

2. $S \backslash S^{(\mathfrak{P},\gamma)}$ has a $\sigma$-**discrete closed** cover $\Sigma \subset \mathfrak{P}$.

On the other hand, for the members of $\mathfrak{P}$, we have that

**Fact 2.** $\mathcal{D} \subset \mathfrak{P}$ discrete in $\bigcup \mathcal{D} \quad \Longrightarrow \quad \bigcup \mathcal{D} \in \mathfrak{P}$.

In view of our next arguments, let us make the following

**Assumption.** $\qquad S^{(\mathfrak{P},\gamma)} \in \mathfrak{P}$.

As a result, we now get that

**Conclusion 3.** There exists a countable cover $\mathcal{F}$ of $S$ with $\mathcal{F} \subset \mathcal{P}(S) \cap \mathfrak{P}$.

**Conclusion 4.** $A \times S \overset{\text{well}}{\hookrightarrow} X \times S$.

Here, $A \times S \overset{\text{well}}{\hookrightarrow} X \times S$ if $A \times S$ is completely separated from any zero-set of $X \times S$ which doesn't meet $A \times S$. To involve Conclusion 4, we also need the following *weak embedding* properties:

<u>*$C_1$-embedding*</u>: A subset $B$ of $Y$ is *$C_1$-embedded* in $Y$, or briefly $B \overset{C_1}{\hookrightarrow} Y$, if $F \overset{\text{well}}{\hookrightarrow} Y$ for every zero-set $F$ of $B$. That is, for any zero-set $F$ of $B$ and any zero-set $Z$ of $Y$, with $Z \cap F = \emptyset$, there exists a zero-set $Z_F$ of $Y$ such that $F \subset Z_F$ and $Z_F \cap Z = \emptyset$.

<u>*$CU$-embedding*</u>: A subset $B$ of $Y$ is *$CU$-embedded* in $Y$, or briefly $B \overset{CU}{\hookrightarrow} Y$, if for any zero-set $F$ of $B$ and any zero-set $Z$ of $Y$, with $Z \cap F = \emptyset$, there exists a zero-set $Z_F$ of $Y$ such that $F \subset Z_F$ and $Z_F \cap Z \cap B = \emptyset$.

The relations between our weak-embedding properties could be now summarized into the following diagram.

**Observation 5.**

$$\begin{array}{ccc} C^* & & U^\omega \\ & \searrow \;\; \swarrow & \\ & C_1 = CU + \text{well} & \end{array}$$

Then, by Conclusion 4, we have

**Conclusion 6.** $A \times S \overset{C_1}{\hookrightarrow} X \times S$.

According to Conclusion 3, this implies

**Final Conclusion.** $S \in \mathfrak{P}$.

The so obtained contradiction provides the following result which accomplishes the first step of the proof of our theorem.

**Theorem A.** *$M \in \mathfrak{P}$ provided $S \in \mathfrak{P}$ for any weight-homogeneous and nowhere locally compact $S \in \mathcal{P}(M)$.*

SECOND STEP: Separating the factors

NOTATIONS: For sets $D$ and $R$, let $R^D$ denote all maps from $D$ to $R$, and $2^R$ — all subsets of $R$. For cardinals $\kappa$ and $\mu$, let $\kappa^{<\mu} = \bigcup\{\kappa^\delta : \delta < \mu\}$. For reasons of convenience, we regard $\kappa^0$ as the singleton $\{\emptyset\}$. To every $\sigma \in \kappa^\delta$ and $\alpha < \kappa$ we associate another map $\sigma\hat{\ }\alpha \in \kappa^{\delta+1}$ defined by $\sigma\hat{\ }\alpha|\delta = \sigma$ and $\sigma\hat{\ }\alpha(\delta) = \alpha$. Also, to every $\mathcal{H} : T \to \left(2^R\right)^D$ we associate another map $\langle\mathcal{H}, D\rangle : T \to 2^R$ defined by

$$\langle\mathcal{H}, D\rangle(t) = \bigcup \mathcal{H}[t](D) \quad \forall\, t \in T.$$

Finally, for a space $Y$, we shall use $\operatorname{coz}(Y)$ to denote the collection of all *cozero-sets* of $Y$ and $\operatorname{zero}(Y)$ for that of all *zero-sets* of $Y$.

CONCEPTS:

*Monotone decreasing map*: $\mathcal{H} : \kappa^{<\omega} \to (2^R)^D$ if $\mathcal{H}[\sigma\hat{\ }\alpha](D)$ refines $\mathcal{H}[\sigma](D)$ for every $\sigma \in \kappa^{<\omega}$ and $\alpha < \kappa$.

*Sieve*: $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(Y)$ if $\mathcal{S}(\emptyset) = Y$ and $\mathcal{S}(\sigma) = \bigcup\{\mathcal{S}(\sigma\hat{\ }\alpha) \ : \alpha < \kappa\}$ for every $\sigma \in \kappa^{<\omega}$.

*Strong Sieve*: $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(Y)$ if $\mathcal{S}$ is a sieve such that $\emptyset \notin \mathcal{S}(\kappa^{<\omega})$, each family $\mathcal{S}(\kappa^n)$, $n < \omega$, is a locally finite in $Y$ and, whenever $y \in \bigcap\{\mathcal{S}(t|n) : n < \omega\}$ for some $t \in \kappa^\omega$, the collection $\mathcal{S}(t|n)$, $n < \omega$, stands for a local base at $y$ in $Y$.

*$\mathcal{S}$-free map*: $\mathcal{G} : \kappa^{<\omega} \to (2^Y)^\kappa$, where $\mathcal{S}$ is a map $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$, if for every $t \in \kappa^\omega$ we have that $\bigcap\{\operatorname{cl}_Y(\langle\mathcal{G}, \kappa\rangle(t|n)) \times \mathcal{S}(t|n) : n < \omega\} = \emptyset$.

*Expansion*: $\mathcal{H} : \kappa^{<\omega} \to (2^X)^\kappa$ of $\mathcal{G} : \kappa^{<\omega} \to (2^Y)^\kappa$, where $Y \subset X$, if $\mathcal{G}[\sigma](\alpha) = \mathcal{H}[\sigma](\alpha) \cap Y$ whenever $\sigma \in \kappa^{<\omega}$ and $\alpha < \kappa$.

The second step of the proof of our theorem reads now as follows.

**Theorem B.** *Under the conditions of the main theorem, let, in addition, $M$ be weight homogeneous and nowhere locally compact. Also, let $w(M) = \kappa$. Then, the following conditions are equivalent.*

(a) $A \times M \overset{C^*}{\hookrightarrow} X \times M$

(b) *Whenever $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$ is a strong sieve, every monotone decreasing and $\mathcal{S}$-free map $\mathcal{G} : \kappa^{<\omega} \to \operatorname{coz}(A)^\kappa$ has a monotone decreasing and $\mathcal{S}$-free expansion $\mathcal{G} : \kappa^{<\omega} \to \operatorname{coz}(X)^\kappa$.*

(c) $A \times M \overset{P^\lambda}{\hookrightarrow} X \times M$

Here is a brief scheme of (a) $\Longrightarrow$ (b). Suppose that $\mathcal{G} : \kappa^{<\omega} \to \operatorname{coz}(A)^\kappa$ and $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$ are as in (b). Then, the statement that $\mathcal{G}$ is an $\mathcal{S}$-free map becomes equivalent to the statement that the family $\{\langle \mathcal{G}, \kappa\rangle(\sigma) \times \mathcal{S}(\sigma) : \sigma \in \kappa^{<\omega}\}$ is locally finite in $A \times M$. The last becomes "almost" equivalent to the existence of $F^0_{(\mathcal{G},\mathcal{S})}, F^1_{(\mathcal{G},\mathcal{S})} \in \operatorname{zero}(A \times M)$ such that $F^0_{(\mathcal{G},\mathcal{S})} \cap F^1_{(\mathcal{G},\mathcal{S})} = \emptyset$. However, by (a), $A \times M \overset{C^*}{\hookrightarrow} X \times M$. Hence, there are $Z^0_{(\mathcal{H},\mathcal{S})}, Z^1_{(\mathcal{H},\mathcal{S})} \in \operatorname{zero}(X \times M)$ such that

$$F^i_{(\mathcal{G},\mathcal{S})} \subset Z^i_{(\mathcal{H},\mathcal{S})},\ i < 2, \quad \text{and} \quad Z^0_{(\mathcal{H},\mathcal{S})} \cap Z^1_{(\mathcal{H},\mathcal{S})} = \emptyset.$$

Relying on the "almost" equivalence mentioned above, these two zero-sets of $X \times M$ yield a monotone decreasing and $\mathcal{S}$-free expansion $\mathcal{H} : \kappa^{<\omega} \to \operatorname{coz}(X)^\kappa$ of $\mathcal{G}$.

Here is also a brief scheme of (b) $\Longrightarrow$ (c). This implication is based on the following chain of arguments.

**Fact 1.** There exists a strong sieve $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$ on $M$ such that

$$\mathcal{S}_n(z) = \bigcup\{\mathcal{S}(\sigma) : \sigma \in \kappa^n \ \&\ z \in \operatorname{cl}_M(\mathcal{S}(\sigma))\},\ n < \omega,$$

constitute a local base at $z$ for every $z \in M$.

A CONCEPT MORE: Let $\mathbb{I} = [0,1]$.

*Sieve partition of unity*: $\xi : \kappa^{<\omega} \to C(M, \mathbb{I})$, or a function version of strong sieve, if $\xi[\emptyset]$ is the constant function on $M$ with the value of 1, and $\xi[\sigma] = \sum\{\xi[\sigma\hat{\ }\alpha] : \alpha < \kappa\}$ for every $\sigma \in \kappa^{<\omega}$.

**Fact 2.** For every strong sieve $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$ there exists a sieve-partition of unity $\xi : \kappa^{<\omega} \to C(M, \mathbb{I})$ such that $\operatorname{supp}(\xi[\sigma]) \subset \mathcal{S}(\sigma)$ for every $\sigma \in \kappa^{<\omega}$.

Let $(Y, \|.\|)$ be a Banach space, and let $f : A \times M \to Y$ be a continuous map. The statement of (c) becomes now equivalent to the existence of a continuous map $g : X \times M \to Y$ with $g|A \times M = f$. Towards this end, for every space $T$ we shall associate a map $\Delta_T$

$$T \quad \longrightarrow \quad \Delta_T : C(T \times M, Y) \to C(T, Y)^{\kappa^{<\omega}}$$

that defines into the following manner. Let $\mathcal{S} : \kappa^{<\omega} \to \operatorname{coz}(M)$ be a strong sieve on $M$ as in Fact 1. Take a dense $D \subset M$ with $|D| = \kappa$, and then define a map $\theta : \kappa^{<\omega} \to M$ by $\theta(\alpha) \in D \cap \mathcal{S}(\alpha)$ for every $\alpha \in \kappa^{<\omega}$. Finally, our $\Delta_T$ is defined by $\Delta_T(h)[\sigma](x) = h(x, \theta(\sigma))$ whenever $h \in C(T \times M, Y)$, $\sigma \in \kappa^{<\omega}$ and $x \in T$.

The correspondence $\Delta_T$ is "nice" invertible on the image of $C(T \times M, Y)$ under $\Delta_T$. That is, one could restore in full $h \in C(T \times M, Y)$ relying only on $\Delta_T(h)$. Namely, let $\xi : \kappa^{<\omega} \to C(M, \mathbb{I})$ be a sieve partition of unity on $M$ as in Fact 2 applied to $\mathcal{S}$. Then,

$$(*) \qquad h = \lim_{n\to\infty} \sum\{\xi[\sigma] \cdot \Delta_T(h)[\sigma] : \sigma \in \kappa^n\}.$$

The idea of (b) $\Rightarrow$ (c) could be now stated in the following abstract setting. To the map $f$ we associate the corresponding one $\Phi = \Delta_A(f) : \kappa^{<\omega} \to C(A, Y)$. In this way, the correspondence $\Delta_T$ transforms our extension problem to an extension problem for $\Phi$. Namely, it is now sufficient to find $\Gamma : \kappa^{<\omega} \to C(X, Y)$ subject to the following

<u>**Extension Condition:**</u>

$$\textbf{(EC)} \qquad \Gamma[\sigma]\Big|A = \Phi[\sigma], \quad \text{for every } \sigma \in \kappa^{<\omega};$$

<u>**Continuity Condition:**</u>

$$\textbf{(CC)} \qquad \Gamma \in \Delta_X(C(X \times M, Y)).$$

If one could deal with this last problem, then merely $g = \Delta_X^{\leftarrow}(\Gamma) \in C(X \times M, Y)$ will be the required extension of $f$. Turning to this, let us observe that

$$A \overset{P\lambda}{\hookrightarrow} X \quad \Longrightarrow \quad \text{“many” solutions of } \textbf{(EC)}$$

$$???????? \quad \Longrightarrow \quad \text{at least one solution of } \textbf{(CC)}$$

To discover the nature of **(CC)** we call in use $(*)$ and thus we get the following its more concrete setting:

$$\textbf{(CC)}^* \quad \lim_{n\to\infty} \sum\{\xi[\sigma] \cdot \Gamma[\sigma] : \sigma \in \kappa^n\} \in C(X \times M, Y).$$

We are now ready for the final realization of this implication. Namely, the hidden property "???????" becomes the **controlled** extending of monotone decreasing $\mathcal{S}$-free maps. That is, just these maps will take care about the control on **(CC)**. Briefly, to the map $\Phi$ we associate a sequence $\{\mathcal{F}_\ell : \ell < \omega\}$ of monotone decreasing and $\mathcal{S}$-free maps $\mathcal{F}_\ell : \kappa^{<\omega} \to \operatorname{coz}(A)^\kappa$. According to (b), each $\mathcal{F}_\ell$ admits a monotone decreasing and $\mathcal{S}$-free expansion $\mathcal{G}_\ell : \kappa^{<\omega} \to \operatorname{coz}(X)^\kappa$.

The fact that $\Phi = \Delta_A(f)$ could be now stated as

$$\ell < \omega, \quad m \le n < \omega \quad \& \quad \sigma \in \kappa^n$$

$$\Downarrow$$

$$\|\Phi[\sigma](x) - \Phi[\sigma|m](x)\| \le \frac{1}{2^{\ell+1}} \quad \forall x \in A \setminus \langle \mathcal{F}_\ell, \kappa\rangle(\sigma|m)$$

Relying on this, we finally construct $\Gamma$ just satisfying the same condition, i.e. such that

**(CC)$^{**}$**

$$\ell \leq m \leq n < \omega \quad \& \quad \sigma \in \kappa^n$$

$$\Downarrow$$

$$\|\Gamma[\sigma](x) - \Gamma[\sigma|m](x)\| \leq \frac{1}{2^{\ell+1}} \quad \forall x \in X \setminus \langle \mathcal{G}_\ell, \kappa \rangle (\sigma|m)$$